4.4 – Linear Independence

Definition: If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a set of two or more vectors in a vector space V, then S is said to be a **linearly independent set** if no vector in S can be expressed as a linear combination of the others. A set that is not linearly independent is said to be **linearly dependent**. If S has only one vector, we will agree that it is linearly independent if and only if that vector is nonzero.

$\{\hat{i}, \hat{j}\} = \{\vec{e}_1, \vec{e}_2\}$ is linearly independent in $\mathbb{R}^2$.

$\{\hat{i}, \hat{j}, (2,3)\}$ is not, since $2\hat{i} + 3\hat{j} = (2,3)$

$\hat{i} = (1,0),\ \hat{j} = (0,1)$ $\qquad (a,0)+(0,b) = (0,0)$

$a\hat{i} + b\hat{j} = \vec{0}$ $\qquad 0\hat{i} + 0\hat{j} = \vec{0}$ ✓

Theorem 4.4.1 A nonempty set $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_r\}$ in a vector space V is linearly independent if and only if the only coefficients satisfying the vector equation $k_1\,\mathbf{v}_1 + k_2\,\mathbf{v}_2 + \ldots + k_r\,\mathbf{v}_r = \mathbf{0}$ are $k_1 = 0,\ k_2 = 0,\ \ldots,\ k_r = 0$.

$\Rightarrow$

$\Leftarrow$

pf: ($\Rightarrow$) Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ be a linearly independent set. Suppose $\exists\ k_1, k_2, \ldots, k_r$ not all zero such that $k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r = \vec{0}$.

If $k_i \neq 0$, then $k_i\vec{v}_i = -k_1\vec{v}_1 - k_2\vec{v}_2 - \cdots - k_r\vec{v}_r$

$$\Rightarrow \vec{v}_i = -\frac{k_1}{k_i}\vec{v}_1 - \frac{k_2}{k_i}\vec{v}_2 - \cdots - \frac{k_r}{k_i}\vec{v}_r,$$ a linear combination of the vectors $\vec{v}_j$, $j \neq i$, contradicting linear independence.

($\Leftarrow$) Suppose $\{\vec{v}_i\}$ is not a linearly independent set. Then $\vec{v}_i = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_r\vec{v}_r$ for some i.

$\Rightarrow \vec{v}_i - c_1\vec{v}_1 - c_2\vec{v}_2 - \cdots - c_r\vec{v}_r = \vec{0}$, contradicting that all the coefficients are zero. ✓

The standard basis vectors for R^n are a linearly independent set.

#3 In each part, determine whether the vectors are linearly independent or are linearly dependent in R^4.

a. $\{(3, 8, 7, -3), (1, 5, 3, -1), (2, -1, 2, 6), (4, 2, 6, 4)\} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$

b. $(3, 0, -3, 6), (0, 2, 3, 1), (0, -2, -2, 0), (-2, 1, 2, 1)$

a) Can we have $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$ if $c_1, \dots, c_4$ are not all zero? That is: Does $A\vec{x} = \vec{0}$ have only the trivial solution, where $A = [\vec{v}_1 \; \vec{v}_2 \; \vec{v}_3 \; \vec{v}_4]$?

$$A = \begin{bmatrix} 3 & 1 & 2 & 4 \\ 8 & 5 & -1 & 2 \\ 7 & 3 & 2 & 6 \\ -3 & -1 & 6 & 4 \end{bmatrix} \qquad \det(A) = \begin{vmatrix} 3 & 1 & 2 & 4 \\ 8 & 5 & -1 & 2 \\ 7 & 3 & 2 & 6 \\ -3 & -1 & 6 & 4 \end{vmatrix}$$

$R_4 \to R_4 + R_1$

$$\det(A) = \begin{vmatrix} 3 & 1 & 2 & 4 \\ 8 & 5 & -1 & 2 \\ 7 & 3 & 2 & 6 \\ 0 & 0 & 8 & 8 \end{vmatrix} \overset{\text{it turns out}}{=} 0$$

There is a notrivial solution to $A\vec{x} = \vec{0}$

so the set is dependent.

Row reduction yields

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ So c_4 is free

b) You can verify that det(A) (whose columns are the vectors) is $\neq 0$.

The set is independent.

#5 In each part, determine whether the matrices are linearly independent or dependent.

a. $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ in M_{22}

b. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ in M_{23}

a) $c_1 \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{aligned} c_1 + c_2 + 0 &= 0 \\ 0 + 2c_2 + c_3 &= 0 \\ c_1 + 2c_2 + 2c_3 &= 0 \\ 2c_1 + c_2 + c_3 &= 0 \end{aligned} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Each column has a pivot, so the set is independent.

b) is independent by inspection (we looked at it).

$(0, 21, 7, -7) + (-12, 0, -10, -2) + (12, -21, 3, 9)$

#9

a. Show that the three vectors $\mathbf{v}_1 = (0,\ 3,\ 1,\ -1)$, $\mathbf{v}_2 = (6,\ 0,\ 5,\ 1)$, $\mathbf{v}_3 = (4,\ -7,\ 1,\ 3)$ form a linearly dependent set in R^4.

b. Express each vector in part (a) as a linear combination of the other two.

$$\begin{bmatrix} 0 & 6 & 4 \\ 3 & 0 & -7 \\ 1 & 5 & 1 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \rightarrow \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -7/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

a) Since there are only 2 pivots, the set is dependent.

b) so $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ if $c_1 = \frac{7}{3}c_3$ and $c_2 = -\frac{2}{3}c_3$. Let $c_3 = 3$. Then $c_1 = 7$, $c_2 = -2$, and

$$7\vec{v}_1 - 2\vec{v}_2 + 3\vec{v}_3 = \vec{0}$$

$\vec{v}_1 = \frac{2}{7}\vec{v}_2 - \frac{3}{7}\vec{v}_3$, etc.

#14 In each part, let $T_A : R^3 \to R^3$ be multiplication by A and let $\mathbf{u}_1 = (1,\ 0,\ 0)$, $\mathbf{u}_2 = (2,\ -1,\ 1)$, and $\mathbf{u}_3 = (0,\ 1,\ 1)$. Determine whether the set $\{T_A(\mathbf{u}_1),\ T_A(\mathbf{u}_2),\ T_A(\mathbf{u}_3)\}$ is linearly independent in R^3.

a. $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & 2 & 0 \end{bmatrix}$ $\quad T_A(\vec{u}_2) = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -3 \\ 2 & 2 & 0 \end{bmatrix}$

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + 1\vec{0} = \vec{0}$

Theorem 4.4.2

a) A set with finitely many vectors that contains **0** is linearly dependent.

b) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

Theorem 4.4.3 Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in R^n. If $r > n$, then S is linearly dependent.

This is because a matrix equation for the associated homogeneous system has more columns than rows and so contains free variables.

Definition: If $\mathbf{f}_1 = f_1(x),\ \ \mathbf{f}_2 = f_2(x),\ \ldots,\ \mathbf{f}_n = f_n(x)$ are functions that are $n-1$ times differentiable on the interval $(-\infty,\ \infty)$, then the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the **Wronskian** of $f_1,\ f_2, \ldots,\ f_n$.

Theorem 4.4.4 If the functions $\mathbf{f}_1,\ \mathbf{f}_2,\ \ldots,\ \mathbf{f}_n$ have $n-1$ continuous derivatives on the interval $(-\infty,\ \infty)$, and if the Wronskian of these functions is not identically zero on $(-\infty,\ \infty)$, then these functions form a linearly independent set of vectors in $C^{(n-1)}(-\infty,\ \infty)$.

#19 Use the Wronskian to show that the following sets of vectors are linearly independent.

a. $1,\ x,\ e^x$

b. $1,\ x,\ x^2$

a) $\vec{f}_1 = f_1(x) = 1,\ \vec{f}_2 = f_2(x) = x,\ \ \vec{f}_3 = f_3(x) = e^x$

$$W(f_1, f_2, f_3) = \begin{vmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix} = e^x \neq 0 \quad \forall x.$$

so $\{1,\ x,\ e^x\}$ is independent.

$$\begin{vmatrix} 1 & \sin^2 x & \cos 2x \\ 0 & \sin 2x & -2\sin 2x \\ 0 & 2\cos 2x & -4\cos 2x \end{vmatrix} = 1 \begin{vmatrix} \sin 2x & -2\sin 2x \\ 2\cos 2x & -4\cos 2x \end{vmatrix} \equiv 0$$